

The power of linearity and superposition I
The time domain: Convolution and the Impulse Response

Convolution can be used to calculate the response of a system to arbitrary inputs by using the impulse response of a system.

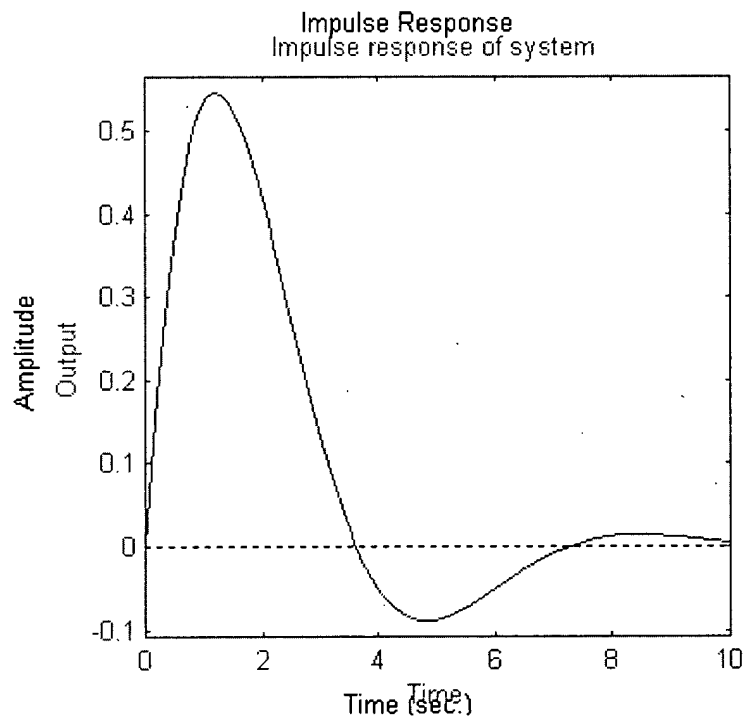
As an example consider the system described by the differential equation:

$$\ddot{y}(t) + \dot{y}(t) + y(t) = f(t)$$

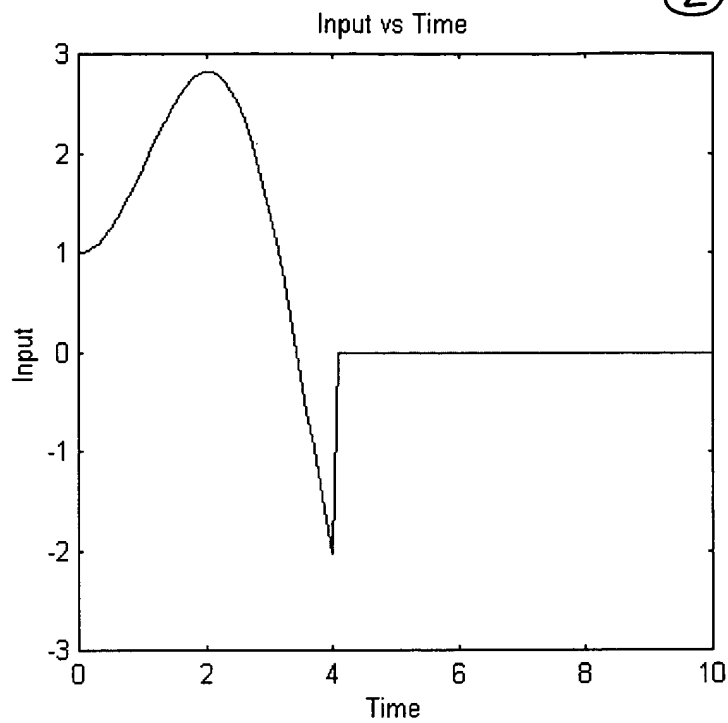
where $y(t)$ is the output, and $f(t)$ is the input. The impulse response of this system, $h(t)$, can be shown to be equal to

$$h(t) = \left(\frac{1}{\sqrt{3}} \right) e^{-0.5t} \sin\left(\sqrt{\frac{3}{4}}t\right), \quad t > 0 \quad (\text{from Laplace Transform Table})$$

The impulse response is shown below.



You have learned techniques for finding $y(t)$ if $f(t)$ is a well defined input such as a step, impulse or sinusoid. Convolution allows you to determine the response to more complex inputs like the one shown below.

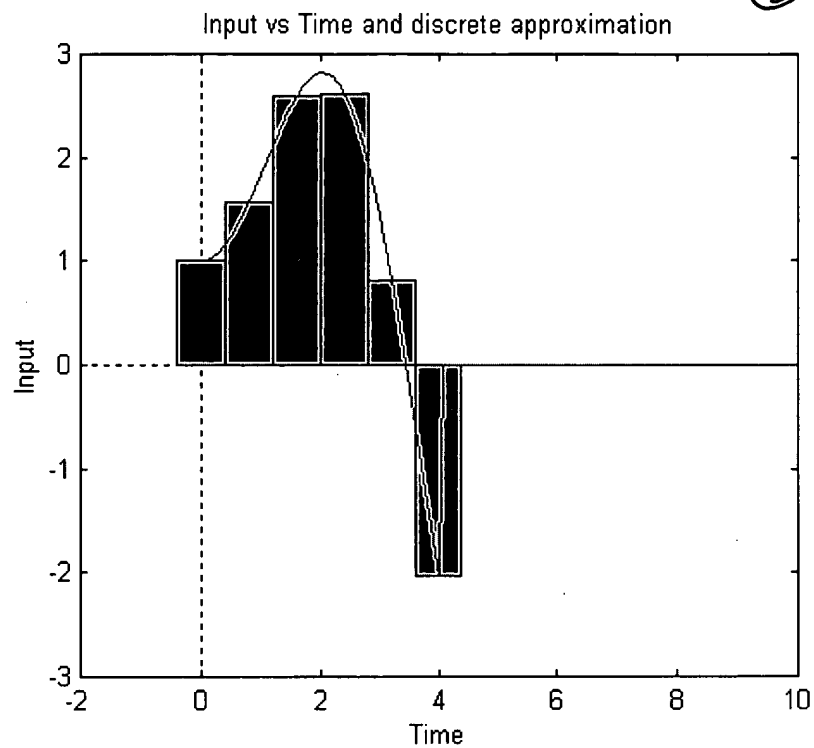


There are several ways to understand how convolution works. First convolution will be developed in an approximate form as the sum of impulse responses. This presentation is useful for an intuitive understanding of the convolution process. After the approximate form is developed, the [exact analytic form of convolution](#) is given. You can also link to an example of the [convolution integral in action](#), and to a [less physical \(more mathematical\) derivation](#).

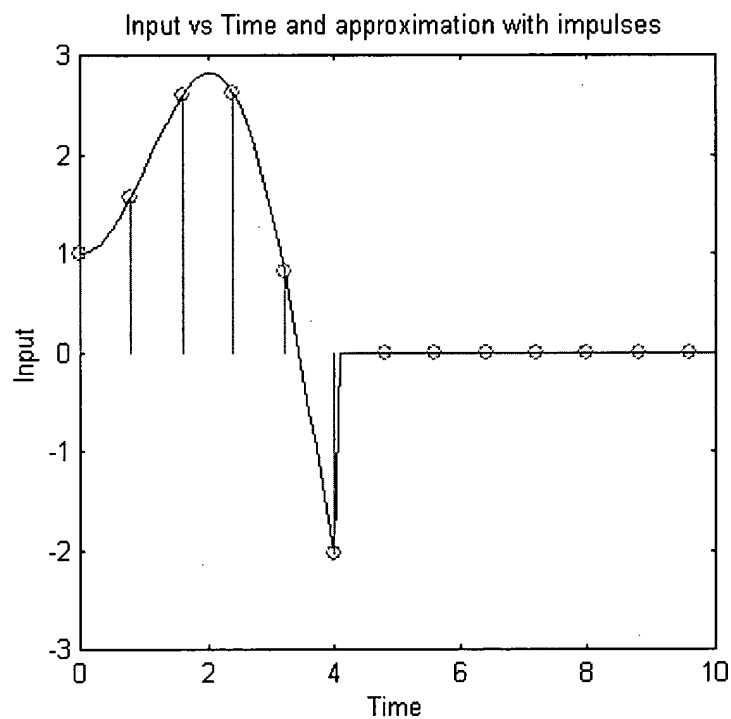
The convolution as a sum of impulse responses.

(the Matlab script, [Convolution.m](#), was used to create all of the graphs in this section).

To understand how convolution works, we represent the continuous function shown above by a discrete function, as shown below, where we take a sample of the input every 0.8 seconds.

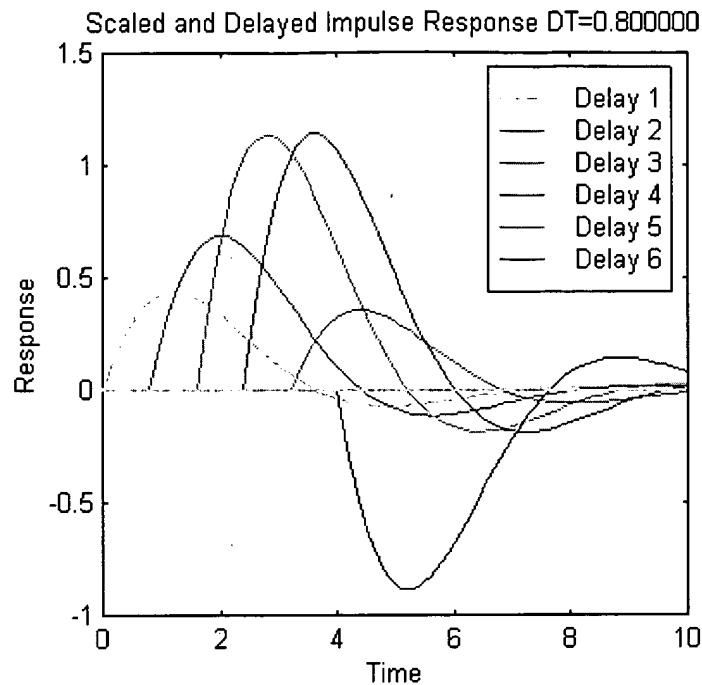


The approximation can be taken a step further by replacing each rectangular block by an impulse as shown below. The area of each impulse is the same as the area of the corresponding rectangular block.



The superposition theorem states that the response of the system to the string of impulses is just the sum of the response to the individual impulses. The response of the system to the individual impulses is shown below.

(4)



Lets look carefully at this graph and note the following:

- The response of the system to a unit impulse, $\delta(t)$, is $h(t)$.
- The response of the system to a unit impulse delayed by time $i \cdot \Delta T$, $\delta(t - i \cdot \Delta T)$, is $h(t - i \cdot \Delta T)$.
- The response of the system to an impulse of area A delayed by time $i \cdot \Delta T$, is $A \cdot h(t - i \cdot \Delta T)$.
- The height of the function at a time $t = i \cdot \Delta T$ is $f(i \cdot \Delta T)$.
- The area of the impulse at $t = i \cdot \Delta T$ is $f(i \cdot \Delta T) \cdot \Delta T$.
- The delayed and shifted impulse response is given by $f(i \cdot \Delta T) \cdot \Delta T \cdot h(t - i \cdot \Delta T)$
- Therefore the response of the system to a series of delayed impulses, as shown above is

$$y(t) = \sum_{i=0}^{\infty} (f(i \cdot \Delta T) \Delta T) h(t - i \cdot \Delta T)$$

In this equation

- $y(t)$ is the output
- $i \cdot \Delta T$ is the time delay of each impulse
- $(f(i \cdot \Delta T) \cdot \Delta T)$ is the area of the i^{th} impulse
- if you take the limit as $\Delta T \rightarrow 0$, the summation yields the convolution integral (with $i \cdot \Delta T = \tau$, $\Delta T = d\tau$)

$$y(t) = \int_0^{+\infty} f(\tau) h(t - \tau) d\tau$$

This is the Convolution Theorem.

- The integral is often presented with limits of positive and negative infinity:

$$y(t) = \int_{-\infty}^{+\infty} f(\tau) h(t - \tau) d\tau$$

For our purposes the two integrals are equivalent because $f(\tau)=0$ for $\tau < 0$

- The form of the integral we will most often use is

$$y(t) = \int_0^t f(\tau) h(t - \tau) d\tau$$

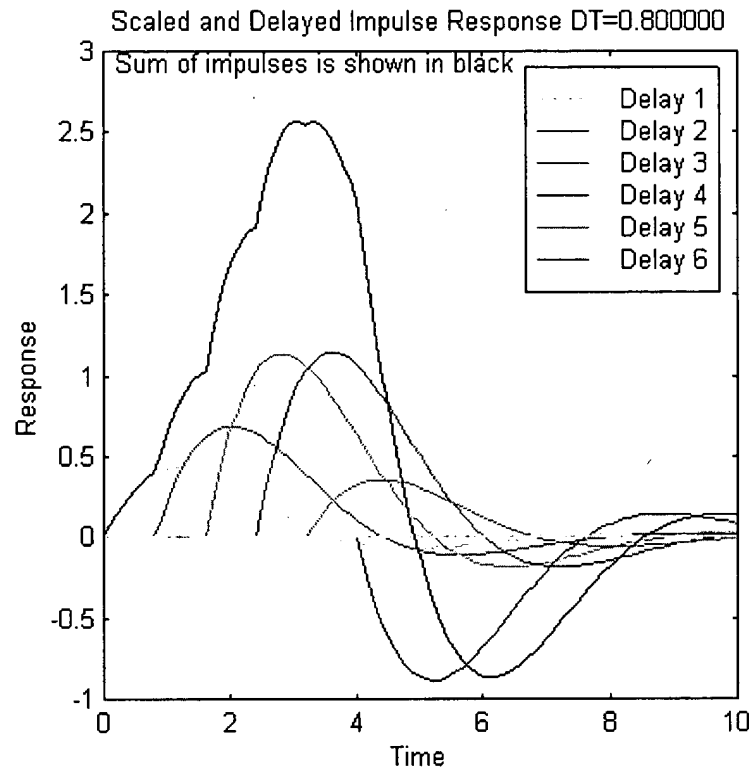
This form is also equivalent because $h(t-\tau)=0$ for $t-\tau<0$ (or $h(t-\tau)=0$ for $\tau>t$). This makes the limits of integration effectively 0 and t .

- o The arguments in the integral can also be switched to give two equivalent forms of the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)f(t-\tau)d\tau$$

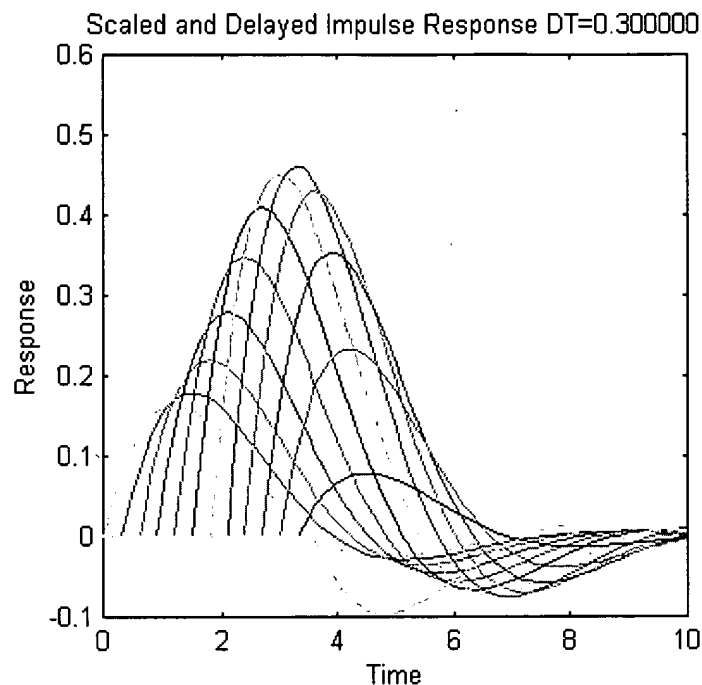
This equation merely states that the output is equal to the sum of the responses from the individual impulses. Another (more mathematical) derivation of the convolution integral is given [here](#).

The summed response is shown in the graph below, along with the individual responses.

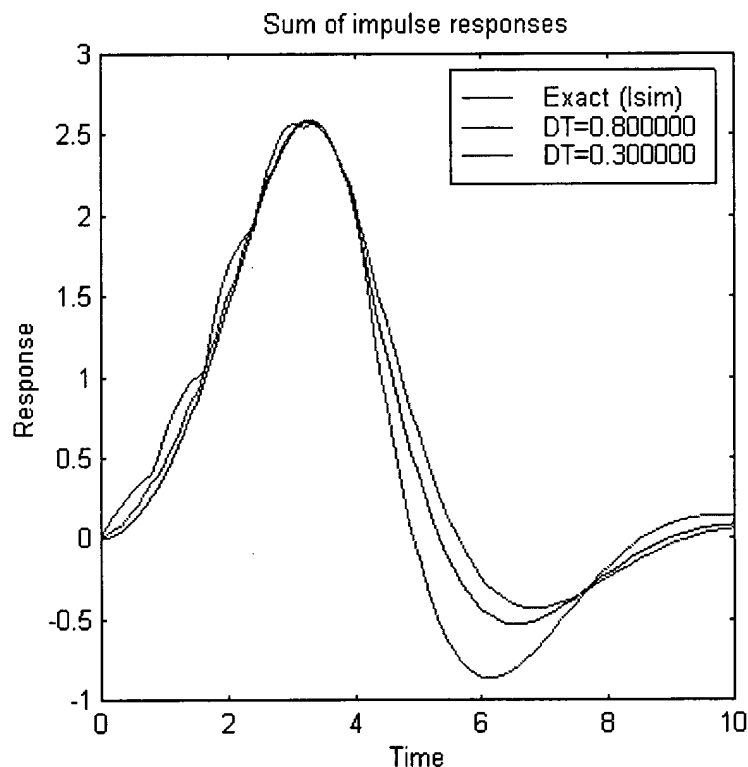


Obviously if you decrease the accuracy of the approximation should improve. The graph below shows the impulse responses if we let $\Delta T=0.3$ seconds (instead of 0.8 seconds as shown above).

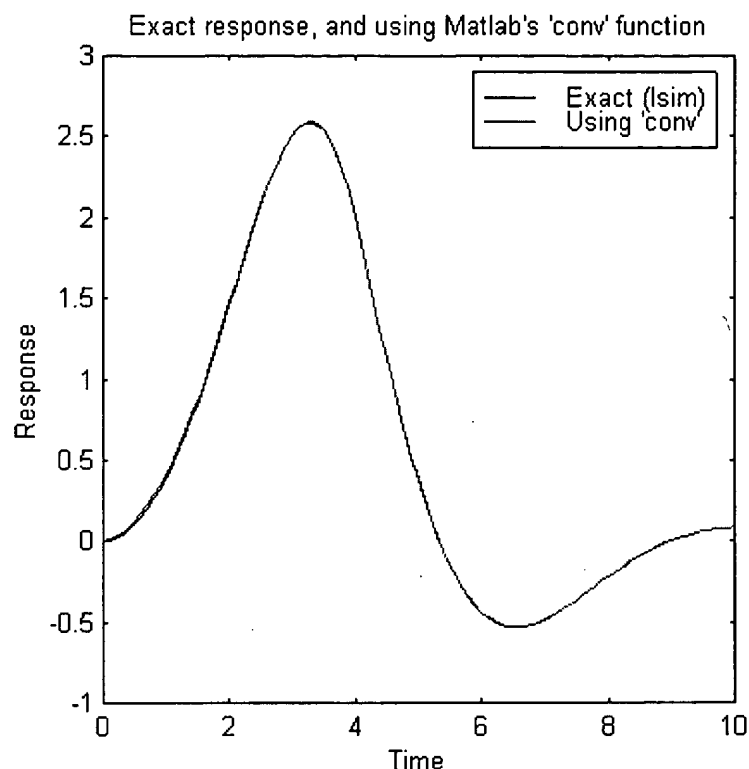
6



The graph below shows our approximate response, for $\Delta T=0.8$ and $\Delta T=0.3$, along with the exact response (the "exact" response is actually a numerical approximation, as calculated by Matlab).



The graph below shows the "exact" response along with the response calculated using Matlab "conv" (convolve) function. Type "help conv" at the Matlab prompt to see how it works. Examine the script, [convolution.m](#), which generated all these graphs for an example of how to use the function.



Hopefully this background gives you some physical insight into how convolution can be used to find the response of the system to an arbitrary input.

An analytic description of convolution

To describe the analytic form of convolution consider the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

or equivalently

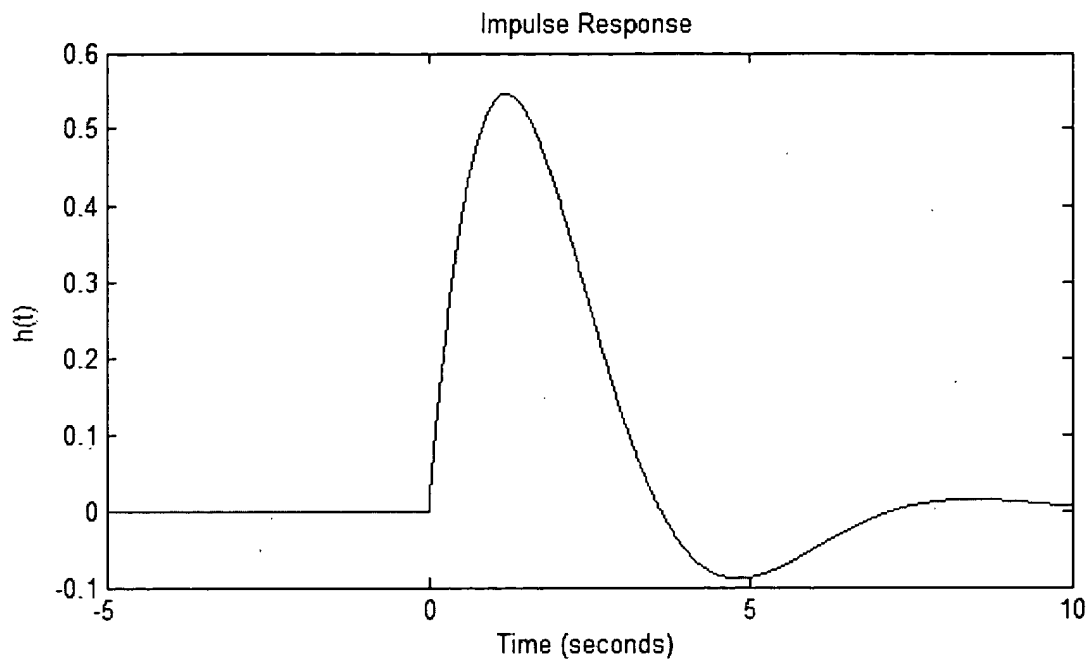
$$y(t) = \int_0^{+\infty} f(\tau)h(t-\tau)d\tau$$

if $h(t)=0$, $t<0$ and $f(t)=0$, $t<0$.

Consider the system discussed previously with the impulse response described by:

$$h(t) = \left(\frac{1}{\sqrt{\frac{3}{4}}} \right) e^{-0.5t} \sin\left(\sqrt{\frac{3}{4}}t\right), \quad t > 0$$

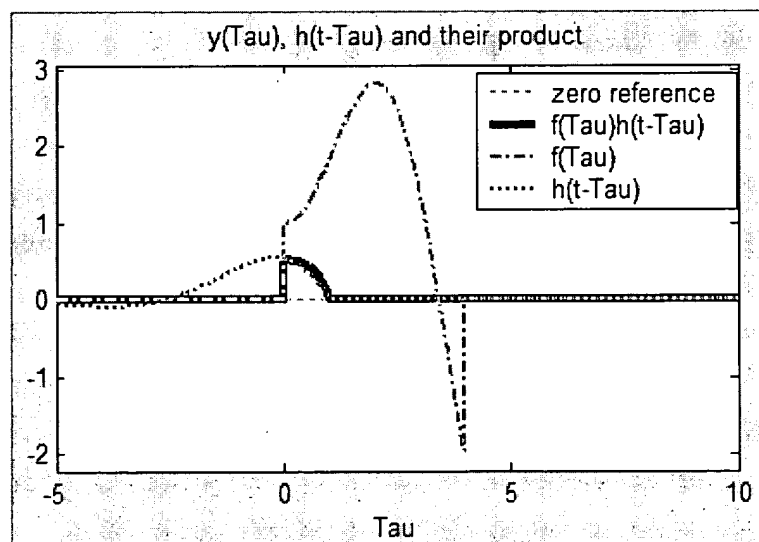
as shown below



To find the response of the system, $y(t)$, to an input, $f(t)$, at $t=1$ sec we need to evaluate the integral

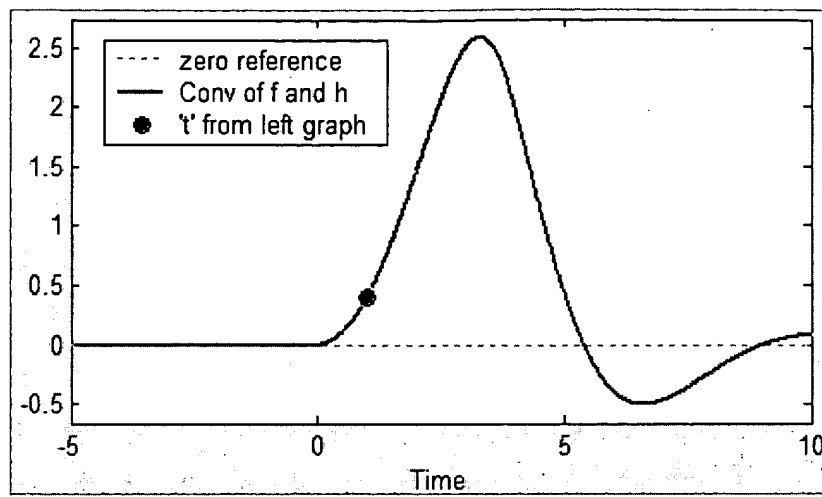
$$y(t) = \int_0^{+\infty} f(\tau)h(t-\tau)d\tau$$

with $t=1$. To visualize this process consider the product $f(\tau)h(t-\tau)$. A plot of $f(\tau)$ yields the input. A plot of $h(t-\tau)$ yields a time-reversed version of the impulse response with a time delay of t (1 second in this case). These are shown below along with the product of the two functions (solid black line). The function $h(t-\tau)$ is plotted in magenta and doesn't look much like the function above -- but note the different vertical scales.

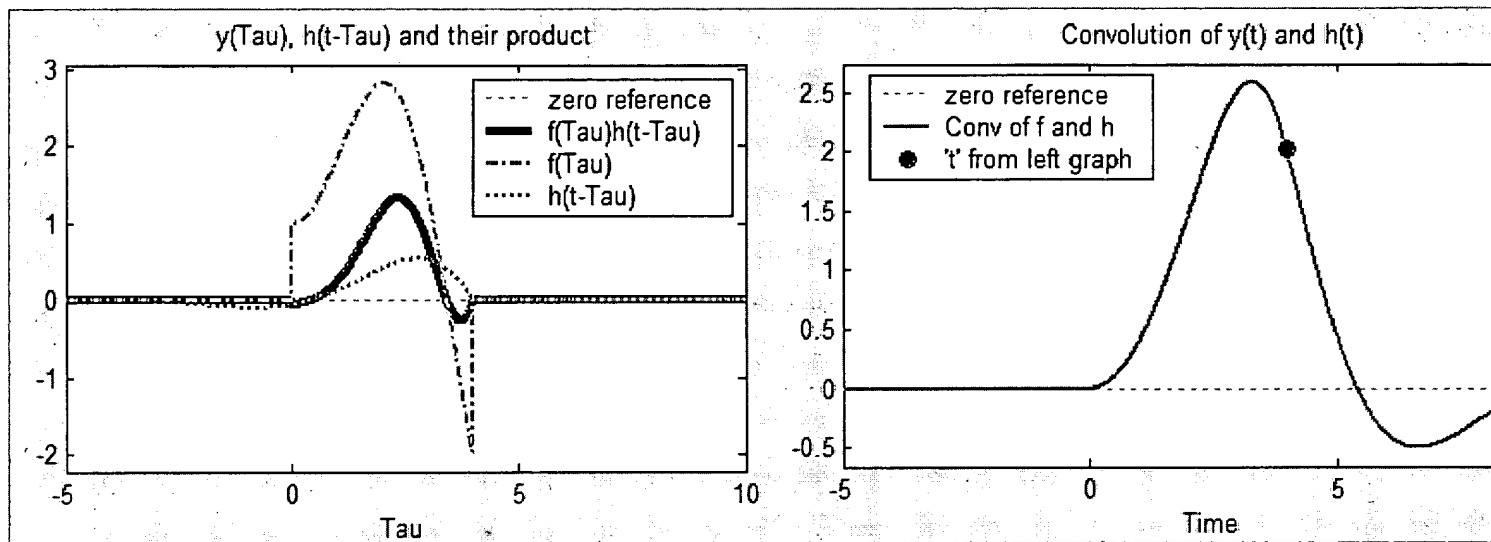


The next graph shows the result of the integration for all time, with a black dot at $t=1$.

9



As t is increased further the function $h(t-\tau)$ shifts to the right. The multiplication and integration can be evaluated for each value of t to yield the system response $y(t)$. The graphs below are similar to the ones above, except that $t=4$.



To see an animation of this shifting, multiplying and integrating process, [click here](#).

Using the convolution integral it is possible to calculate the output, $y(t)$, of any linear system given only the input, $f(t)$, and the impulse response, $h(t)$. However, this integration is often difficult, so we won't often do it explicitly. Later you will learn a technique that vastly simplifies the convolution process.

Some more examples

To see some more animations of convolutions, [click here](#).

Another demonstration

Another excellent web-based demonstration of convolution is [The Joy of Convolution](#) at Johns Hopkins.

An Example of the Convolution Integral in Action

This page has given a description of the convolution process, but has not actually gone through the mathematical procedures needed to analytically evaluate the convolution integral. This process is described on another page. [Click here to go to that page](#).

The MatLab code that generated the animations is available [here](#).
